

V bodě  $A = [3, 1]$  určete diferenciál 2. řádu k funkci

$$f(x, y) = \operatorname{arctg}(x - 2y).$$

**Řešení:** Nejprve je potřeba spočítat všechny parciální derivace prvního

$$\begin{aligned}\frac{\partial f(x, y)}{\partial x} &= \frac{\partial \operatorname{arctg}(x-2y)}{\partial x} = \frac{1}{1 + (x - 2y)^2} = (1 + (x - 2y)^2)^{-1}, \\ \frac{\partial f(x, y)}{\partial y} &= \frac{\partial \operatorname{arctg}(x-2y)}{\partial y} = \frac{-2}{1 + (x - 2y)^2} = -2(1 + (x - 2y)^2)^{-1},\end{aligned}$$

a druhého řádu

$$\begin{aligned}\frac{\partial^2 f(x, y)}{\partial x^2} &= \frac{\partial (1 + (x - 2y)^2)^{-1}}{\partial x} = -(1 + (x - 2y)^2)^{-2} (2(x - 2y)) = \frac{-2(x - 2y)}{(1 + (x - 2y)^2)^2}, \\ \frac{\partial^2 f(x, y)}{\partial x \partial y} &= \frac{\partial (1 + (x - 2y)^2)^{-1}}{\partial y} = -(1 + (x - 2y)^2)^{-2} (2(x - 2y) \cdot (-2)) = \frac{4(x - 2y)}{(1 + (x - 2y)^2)^2}, \\ \frac{\partial^2 f(x, y)}{\partial y \partial x} &= \frac{\partial (-2(1 + (x - 2y)^2)^{-1})}{\partial x} = -2(-(1 + (x - 2y)^2)^{-2} (2(x - 2y))) = \frac{4(x - 2y)}{(1 + (x - 2y)^2)^2}, \\ \frac{\partial^2 f(x, y)}{\partial y^2} &= \frac{\partial (-2(1 + (x - 2y)^2)^{-1})}{\partial y} = -2(-(1 + (x - 2y)^2)^{-2} (2(x - 2y)) \cdot (-2)) = \frac{-8(x - 2y)}{(1 + (x - 2y)^2)^2}.\end{aligned}$$

Po této spíše zdlouhavé než těžké části si připomeneme vzorec pro výpočet diferenciálu 2. řádu k  $f(x, y)$ , dosadíme vypočtené a provedeme „pár úprav“ ke zřehlednění výsledku.

$$\begin{aligned}d^2f &= \frac{\partial^2 f(x, y)}{\partial x^2} dx^2 + 2 \frac{\partial^2 f(x, y)}{\partial x \partial y} dx dy + \frac{\partial^2 f(x, y)}{\partial y^2} dy^2, \\ d^2f &= \frac{-2(x - 2y)}{(1 + (x - 2y)^2)^2} dx^2 + \frac{8(x - 2y)}{(1 + (x - 2y)^2)^2} dx dy + \frac{-8(x - 2y)}{(1 + (x - 2y)^2)^2} dy^2 \\ d^2f &= \frac{-2(x - 2y)}{(1 + (x - 2y)^2)^2} (dx^2 - 4 dx dy + 4 dy^2)\end{aligned}$$

Nyní již zbývá jen dosadit bod  $A$ , tj. dosadit  $x = 3$  resp.  $y = 1$  a dále  $dx = (x - 3)$  resp.  $dy = (y - 1)$ , čímž dostaneme

$$\begin{aligned}d^2f &= \frac{-2(x - 2y)}{(1 + (x - 2y)^2)^2} (dx^2 - 4 dx dy + 4 dy^2) \\ d^2f(\mathbf{A}) &= \frac{-2(3 - 2 \cdot 1)}{(1 + (3 - 2 \cdot 1)^2)^2} ((x - 3)^2 - 4(x - 3)(y - 1) + 4(y - 1)^2) \\ d^2f(\mathbf{A}) &= -\frac{1}{2} ((x^2 - 6x + 9) - 4(xy - x - 3y + 3) + 4(y^2 - 2y + 1)) \\ d^2f(\mathbf{A}) &= -\frac{1}{2} (x^2 - 4xy - 2x + 4y^2 + 4y + 1) \\ d^2f(\mathbf{A}) &= -\frac{x^2}{2} + 2xy + x - 2y^2 - 2y - \frac{1}{2}.\end{aligned}$$

A máme hotovo.

**K funkci  $f(x, y)$  určete diferenciál 2. řádu, a to i v bodě  $\mathbf{A}$  pokud je zadán.**

(1)  $f(x, y) = \cos(x - y)$

(12)  $f(x, y) = \operatorname{arctg}(xy)$ .

(2)  $f(x, y) = (x - 2y)^3$

(13)

$f(x, y) = x^2 + 3xy^3 - 4x + 2y + 5, \quad \mathbf{A} = [-1, 2].$

(3)  $f(x, y) = (3x - 2y)^{-1}$ .

(14)

$f(x, y) = 3x^3 - 2x^2y + 5xy^2 - 6x + 3y - 10,$   
 $\mathbf{A} = [1, -1].$

(4)  $f(x, y) = \ln(x^2 - y)$ .

(5)  $f(x, y) = \sin(x - y^2)$ .

(15)

$f(x, y) = (x^2 - 1)y + 5xy - 2x + 3y^2 + 7,$   
 $\mathbf{A} = [-2, 0].$

(6)  $f(x, y) = \ln\left(\frac{x}{y}\right)$ .

(7)  $f(x, y) = x^2 \ln\left(\frac{1}{y}\right), \quad \mathbf{A} = [-1, 1].$

(16)  $f(x, y) = \sqrt{x + 2y}, \quad \mathbf{A} = [-1, 1].$

(8)  $f(x, y) = \ln(x + 2y)$ .

(17)  $f(x, y) = \sin(xy) - \cos(x), \quad \mathbf{A} = [0, 1].$

(9)  $f(x, y) = \frac{x}{y}$ .

(18)  $f(x, y) = (3x^2 - y)^2, \quad \mathbf{A} = [0, -1].$

(10)  $f(x, y) = \ln(x^2 + y^2)$

(19)  $f(x, y) = \frac{x}{2y} - \frac{y}{2x}, \quad \mathbf{A} = [1, -1].$

(11)  $f(x, y) = 3 \cos(2x - 3y + 5)$

(20)  $f(x, y) = \cos(3x - 2y), \quad \mathbf{A} = [2, 3].$

K funkci  $f(x, y)$  určete diferenciál 2. řádu, a to i v bodě  $\mathbf{A}$  pokud je zadán.

$$(1) \quad d^2f = -\cos(x-y) (dx^2 - 2 dx dy + dy^2).$$

$$(11) \quad d^2f = -3 \cos(2x-3y+5) (4 dx^2 - 12 dx dy + 9 dy^2).$$

$$(2) \quad d^2f = 6(x-2y) (dx^2 - 4 dx dy + 4 dy^2).$$

$$(12) \quad d^2f = -\frac{2(xy^3 dx^2 - (1-x^2y^2) dx dy + x^3y dy^2)}{(1+x^2y^2)^2}.$$

$$(3) \quad d^2f = \frac{18 dx^2 - 24 dx dy + 8 dy^2}{(3x-2y)^3}.$$

$$(13) \quad d^2f(\mathbf{A}) = 2x^2 + 72xy - 36y^2 - 140x + 216y - 286.$$

$$(4) \quad d^2f = \frac{-2(x^2+y) dx^2 + 4x dx dy - dy^2}{(x^2-y)^2}.$$

$$(14) \quad d^2f(\mathbf{A}) = 22x^2 - 28xy + 10y^2 - 72x + 48y + 60.$$

$$(5) \quad d^2f = -\sin(x-y^2) dx^2 + 4y \sin(x-y^2) dx dy + \left(-2 \cos(x-y^2) - 4y^2 \sin(x-y^2)\right) dy^2.$$

$$(15) \quad d^2f(\mathbf{A}) = 2xy + 4y + 6y^2.$$

$$(6) \quad d^2f = -\frac{dx^2}{x^2} + \frac{dy^2}{y^2}.$$

$$(16) \quad d^2f(\mathbf{A}) = -\frac{x^2}{4} - xy + \frac{x}{2} - y^2 + y - \frac{1}{4}.$$

$$(7) \quad d^2f(\mathbf{A}) = y^2 + 4xy + 2y - 4x - 3.$$

$$(17) \quad d^2f(\mathbf{A}) = x^2 + 2xy - 2x.$$

$$(8) \quad d^2f = -\frac{dx^2 + 4 dx dy + 4 dy^2}{(x+2y)^2}.$$

$$(18) \quad d^2f(\mathbf{A}) = 12x^2 + 2y^2 + 4y + 2.$$

$$(9) \quad d^2f = \frac{-2y dx dy + 2x dy^2}{y^3}.$$

$$(19) \quad d^2f(\mathbf{A}) = x^2 - 2x - y^2 - 2y.$$

$$(10) \quad d^2f = \frac{2((y^2-x^2) dx^2 - 4xy dx dy + (x^2-y^2) dy^2)}{(x^2+y^2)^2}.$$

$$(20) \quad d^2f(\mathbf{A}) = -9x^2 + 12xy - 4y^2.$$